Structural Properties of Utility Functions
Walrasian Demand

Econ 3030 Fall 2025

Lecture 4

Outline

From Last Class

Definition

The utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow u(\mathbf{x}) \geq u(\mathbf{y}).$$

Theorem (Debreu)

Suppose $X \subseteq \mathbb{R}^n$. A binary relation $\succsim$ on X is complete, transitive, and continuous if and only if it admits a continuous utility representation $u : X \to \mathbb{R}$.

- We are interested in connections between utility functions and preferences.

Structural Properties of Utility Functions

- The main idea is to understand the relation between properties of preferences and characteristics of the utility function that represents them.

Local Non Satiation

Definition

A preference relation $\succsim$ is locally nonsatiated if for all $\mathbf{x} \in X$ and $\varepsilon > 0$, there exists some $\mathbf{y}$ such that $\|\mathbf{y} - \mathbf{x}\| < \varepsilon$ and $\mathbf{y} \succ \mathbf{x}$.

- For any consumption bundle, there is always a nearby bundle that is strictly preferred to it.

Example: The lexicographic preference on $\mathbb{R}^2$ is locally nonsatiated

- Fix (x_1, x_2) and $\varepsilon > 0$.
- Then $(x_1 + \frac{\varepsilon}{2}, x_2)$ satisfies $\|(x_1 + \frac{\varepsilon}{2}, x_2) - (x_1 . x_2)\| < \varepsilon$
- and $(x_1 + \frac{\varepsilon}{2}, x_2) \succ (x_1, x_2)$.

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- For any consumption bundle, there is always a nearby bundle that is strictly preferred to it.

Definition

A utility function $u : X \to \mathbb{R}$ is locally nonsatiated if it represents a locally nonsatiated preference relation $\succsim$; that is, if for every $\mathbf{x} \in X$ and $\varepsilon > 0$, there exists some $\mathbf{y}$ such that $\|\mathbf{y} - \mathbf{x}\| < \varepsilon$ and $u(\mathbf{y}) > u(\mathbf{x})$.

Local Non Satiation and Strict Monotonicity

Proposition

If $\succsim$ is strictly monotone, then it is locally nonsatiated.

Proof.

Let $\mathbf{x}$ be given, and let $\mathbf{y} = \mathbf{x} + \frac{\varepsilon}{n}e$, where $e = (1, ..., 1)$.

- Then we have $y_i > x_i$ for each i.
- Strict monotonicity implies that $\mathbf{y} \succ \mathbf{x}$.
- Note that

$$||\mathbf{y} - \mathbf{x}|| = \sqrt{\sum_{i=1}^{n} \left(\frac{\varepsilon}{n}\right)^2} = \frac{\varepsilon}{\sqrt{n}} < \varepsilon.$$

- Thus $\succsim$ is locally nonsatiated. □

Convex Preferences

Definitions

A preference relation $\succsim$ is

- convex if
$$\mathbf{x} \succsim \mathbf{y} \quad \Rightarrow \quad \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y} \text{ for all } \alpha \in (0,1)$$
- strictly convex if
$$\mathbf{x} \succsim \mathbf{y} \text{ and } \mathbf{x} \neq \mathbf{y} \quad \Rightarrow \quad \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succ \mathbf{y} \text{ for all } \alpha \in (0,1)$$

- Convexity says that taking convex combinations cannot make the decision maker worse off.
- Strict convexity says that taking convex combinations makes the decision maker better off.

Question

- What does convexity imply for the utility function representing $\succsim$?

Definitions

Suppose C is a convex subset of X. A function $f : C \to \mathbb{R}$ is:

- concave if
$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$$
for all $\alpha \in [0,1]$ and $\mathbf{x}, \mathbf{y} \in C$;
- strictly concave if
$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) > \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$$
for all $\alpha \in (0,1)$ and $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \neq \mathbf{y}$;
- quasiconcave if
$$f(\mathbf{x}) \geq f(\mathbf{y}) \Rightarrow f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \geq f(\mathbf{y})$$
for all $\alpha \in [0,1]$;
- strictly quasiconcave if
$$f(\mathbf{x}) \geq f(\mathbf{y}) \text{ and } \mathbf{x} \neq \mathbf{y} \Rightarrow f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) > f(\mathbf{y})$$
for all $\alpha \in (0,1)$.

Convexity and Quasiconcave Utility Functions

- Convexity is equivalent to quasi concavity of the corresponding utility function.

Proposition

If $u : X \to \mathbb{R}$ represents $\succsim$, then:

1. *$\succsim$ is convex if and only if u is quasiconcave;*
2. *$\succsim$ is strictly convex if and only if u is strictly quasiconcave.*

- Convexity of $\succsim$ implies that any utility representation is quasiconcave, but not necessarily concave.

Proof.

Question 3b. Problem Set 2. $\square$

Quasiconcave Utility and Convex Upper Contours

Proposition

Let $\succsim$ be a preference relation on X represented by $u : X \longrightarrow \mathbb{R}$. Then, the upper contour set is a convex subset of X if and only if u is quasiconcave.

Proof.

- Suppose that u is quasiconcave.
  - Fix $\mathbf{z} \in X$, and take any $\mathbf{x}, \mathbf{y} \in \succsim(\mathbf{z})$.
  - Wlog, assume $u(\mathbf{x}) \geq u(\mathbf{y})$, so that $u(\mathbf{x}) \geq u(\mathbf{y}) \geq u(\mathbf{z})$, and let $\alpha \in [0, 1]$.
  - By quasiconcavity of u,
  $$u(\mathbf{z}) \leq u(\mathbf{y}) \leq u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})$$
  so $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{z}$.
  - Hence $\alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ belongs to $\succsim(\mathbf{z})$, proving it is convex.
- Now suppose the better-than set is convex.
  - Let $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in [0, 1]$, and suppose $u(\mathbf{x}) \geq u(\mathbf{y})$.
  - Then $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{y}$, and so $\mathbf{x}$ and $\mathbf{y}$ are both in $\succsim(\mathbf{y})$.
  - Since $\succsim(\mathbf{y})$ is convex (by assumption), then $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y}$.
  - Since u represents $\succsim$,
  $$u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq u(\mathbf{y})$$
  - Thus u is quasiconcave.

□

Convexity and Induced Choices

Proposition

If $\succsim$ is convex, then $C_{\succsim}(A)$ is convex for all convex A.
If $\succsim$ is strictly convex, then $C_{\succsim}(A)$ has at most one element for any convex A.

Proof.

- Let A be convex and $\mathbf{x}, \mathbf{y} \in \mathcal{C}_{\succsim}(A)$.
  - By definition of $C_{\succsim}(A)$, $\mathbf{x} \succsim \mathbf{y}$.
  - Since A is convex: $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A$ for any $\alpha \in [0,1]$.
  - Convexity of $\succsim$ implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y}$.
  - By definition of $C_{\succsim}$, $\mathbf{y} \succsim \mathbf{z}$ for all $\mathbf{z} \in A$.
  - Using transitivity, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y} \succsim \mathbf{z}$ for all $\mathbf{z} \in A$.
  - Hence, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in C_{\succsim}(A)$ by definition of induced choice rule.
  - Therefore, $C_{\succsim}(A)$ is convex for any convex A.
- Now suppose there exists a convex A for which $\left|C_{\succsim}(A)\right| \geq 2$.
  - Then there exist $\mathbf{x}, \mathbf{y} \in C_{\succsim}(A)$ with $\mathbf{x} \neq \mathbf{y}$.
  - Since A is convex, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A$ for all $\alpha \in (0,1)$.
  - Since $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, strict convexity implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succ \mathbf{y}$, but this contradicts the fact that $\mathbf{y} \in C_{\succsim}(A)$. □

Quasi-linear Utility

Definition

The function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasi-linear if there exists a function $v : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that $u(\mathbf{x}, m) = v(\mathbf{x}) + m$.

- We think of the n-th good as money (the numeraire).

Quasi-linear Preferences

Proposition

The preference relation $\succsim$ on $\mathbb{R}^n$ admits a quasi-linear representation if and only if

1. $(\mathbf{x}, m) \succsim (\mathbf{x}, m')$ *if and only if* $m \geq m'$, *for all* $\mathbf{x} \in \mathbb{R}^{n-1}$ *and all* $m, m' \in \mathbb{R}$;
2. $(\mathbf{x}, m) \succsim (\mathbf{x}', m')$ *if and only if* $(\mathbf{x}, m + m'') \succsim (\mathbf{x}', m' + m'')$, *for all* $\mathbf{x} \in \mathbb{R}^{n-1}$ *and* $m, m', m'' \in \mathbb{R}$;
3. *for all* $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{n-1}$, *there exist* $m, m' \in \mathbb{R}$ *such that* $(\mathbf{x}, m) \sim (\mathbf{x}', m')$.

1. If two bundles have identical goods, the consumer prefers the one with more money.
2. Adding (or subtracting) the same monetary amount does not change rankings.
3. Monetary transfers can always be used to achieve indifference.

Proof.

Question 3c. Problem Set 2. □

Quasi-linear Preferences and Utility

Proposition

Suppose that the preference relation $\succsim$ *on* $\mathbb{R}^n$ *admits two quasi-linear representations:* $v(\mathbf{x}) + m$, *and* $v'(\mathbf{x}) + m$, *where* $v, v' : \mathbb{R}^{n-1} \to \mathbb{R}$. *Then there exists* $c \in \mathbb{R}$ *such that* $v'(\mathbf{x}) = v(\mathbf{x}) - c$ *for all* $\mathbf{x} \in \mathbb{R}^{n-1}$.

Proof.

Exercise □

Homothetic Preferences and Utility

- Homothetic preferences are also useful in many applications, in particular for aggregation problems in macroeconomics.

Definition

The preference relation $\succsim$ on X is homothetic if for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \sim \mathbf{y} \Rightarrow \alpha\mathbf{x} \sim \alpha\mathbf{y} \text{ for each } \alpha > 0$$

Homothetic Preferences and Utility

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Definition

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Proposition

The continuous preference relation $\succsim$ on $\mathbb{R}^n$ is homothetic if and only if it is represented by a utility function that is homogeneous of degree 1.

- A function is homogeneous of degree r if $f(\alpha\mathbf{x}) = \alpha^r f(\mathbf{x})$ for any $\mathbf{x}$ and $\alpha > 0$.

Proof.

Question 3d. Problem Set 2. □

Demand Theory

Main Questions

- Suppose the consumer uses her income to purchase goods (commodities) at exogenously given prices:
  - What are the optimal consumption choices?
  - How do they depend on prices and income?

- Typically, we answer this questions solving a constrained optimization problem using calculus.
- That means the utility function must be not only continuous, but also differentiable.
  - Differentiability, however, is not a property we can derive from preferences.
- Sometimes, calculus is not necessary, and we can talk about optimal choices even when preferences are not necessarily represented by a utility function.

Budget Set

- First, we define what a consumer can afford.

Definition

The Budget Set $B(\mathbf{p}, w) \subset \mathbb{R}^n$ at prices $\mathbf{p}$ and income w is the set of all affordable consumption bundles and is defined by

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

- This is the set of consumption bundles the consumer can choose from. She cannot purchase consumption bundles outside of this set.
- Implicit assumptions: goods are perfectly divisible; consumption is non negative; the total price of consumption cannot exceed income; prices are linear. Think of possible violations.

Main Idea

- The optimal consumption bundles are those that are weakly preferred to all other affordable bundles.

Definition

Given a preference relation $\succsim$, the Walrasian demand correspondence $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to$ *all subsets of* $\mathbb{R}^n_+$ is defined by

$$x^*(\mathbf{p}, w) = \{\mathbf{x} \in B(\mathbf{p}, w) : \mathbf{x} \succsim \mathbf{y} \text{ for any } \mathbf{y} \in B(p, w)\}.$$

- By definition, any $\mathbf{x}^* \in x^*(\mathbf{p}, w)$ has the property that

$$\mathbf{x}^* \succsim \mathbf{x} \qquad \text{for any } \mathbf{x} \in B(\mathbf{p}, w).$$

- Walrasian demand equals the induced choice rule for preference relation $\succsim$ and "available set" $B(\mathbf{p}, w)$:

$$x^*(\mathbf{p}, w) = C_{\succsim}(B(\mathbf{p}, w)).$$

- More implicit assumptions: income is non negative; prices are strictly positive.

Walrasian Demand With Utility

- Although we do not need the utility function to exist to define Walrasian demand, if a utility function exists there is an equivalent definition.

Definition

Given a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the Walrasian demand correspondence $x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to$ *all subsets of* $\mathbb{R}^n_+$ is defined by

$$x^*(\mathbf{p}, w) = \arg \max_{\mathbf{x} \in B(\mathbf{p},w)} u(\mathbf{x}) \qquad \text{where} \qquad B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

- From now on, for simplicity, write $\to \mathbb{R}^n_+$ instead of $\to$ *all subsets of* $\mathbb{R}^n_+$
- As before,

$$x^*(\mathbf{p}, w) = C_{\succsim}(B(\mathbf{p}, w)).$$

  and for any $\mathbf{x}^* \in x^*(\mathbf{p}, w)$

$$u(\mathbf{x}^*) \geq u(\mathbf{x}) \qquad \text{for any } \mathbf{x} \in B(\mathbf{p}, w).$$

- We can derive some properties of Walrasian demand directly from assumptions on preferences and/or utility.

Walrasian Demand Is Homogeneous of Degree Zero

Proposition

Walrasian demand is homogeneous of degree zero; that is, for any $\alpha > 0$

$$x^*(\alpha \mathbf{p}, \alpha w) = x^*(\mathbf{p}, w)$$

Proof.

For any $\alpha > 0$,

$$B(\alpha \mathbf{p}, \alpha w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \alpha \mathbf{p} \cdot \mathbf{x} \leq \alpha w\} = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\} = B(\mathbf{p}, w)$$

because α is a scalar

- Since the constraints are the same, the optimal choices must also be the same. $\square$

The Consumer Spends All Her Income

This is sometimes known as Walras' Law for individuals

Proposition (Full Expenditure)

If $\succsim$ is locally nonsatiated , then

$$\mathbf{p} \cdot \mathbf{x} = w \qquad \textit{for any } \mathbf{x} \in x^*(\mathbf{p}, w)$$

Proof.

Suppose not.

- Then there exists an $\mathbf{x} \in x^*(\mathbf{p}, w)$ with $\mathbf{p} \cdot \mathbf{x} < w$.
- By local non-satiation and continuity of the dot product one can find some $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{x}$ and
$$\|\mathbf{y} - \mathbf{x}\| < \varepsilon \text{ with } \varepsilon > 0 \qquad \text{and} \qquad \mathbf{p} \cdot \mathbf{y} \leq w.$$
- This contradicts $\mathbf{x} \in x^*(\mathbf{p}, w)$. □

Walrasian Demand Is Convex

Proposition

If u is quasiconcave, then $x^*(\mathbf{p}, w)$ is convex.
If u is strictly quasiconcave, then $x^*(\mathbf{p}, w)$ is unique.

- $B(\mathbf{p}, w)$ is a convex set (prove this), and $x^*(\mathbf{p}, w) = C_{\succsim}(B(\mathbf{p}, w))$, so we have already proved this (u (stricly) quasiconcave means $\succsim$ (strictly) convex).

Proof.

Suppose $\mathbf{x}, \mathbf{y} \in x^*(\mathbf{p}, w)$ and pick $\alpha \in [0, 1]$.

- Convexity: need to show $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in x^*(\mathbf{p}, w)$.
  - $u(\mathbf{x}) \geq u(\mathbf{y})$, by definition of $x^*(\mathbf{p}, w)$, and $u(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \geq u(\mathbf{y})$, by quasi-concavity
  - $u(\mathbf{y}) \geq u(\mathbf{z})$ for any $\mathbf{z} \in B(\mathbf{p}, w)$ by definition of $x^*(\mathbf{p}, w)$.
  - Thus $u(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \geq u(\mathbf{z})$ for any $\mathbf{z} \in B(\mathbf{p}, w)$ proving $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in x^*(\mathbf{p}, w)$.
- Uniqueness: suppose not, then $\mathbf{x}, \mathbf{y} \in x^*(\mathbf{p}, w)$ and $\mathbf{x} \neq \mathbf{y}$
  - strict quasi-concavity implies $u(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) > u(\mathbf{y})$ for any $\alpha \in (0, 1)$
  - $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in B(\mathbf{p}, w)$, because $B(\mathbf{p}, w)$ is convex, contradicting $\mathbf{y} \in x^*(\mathbf{p}, w)$. □

Walrasian Demand Is Non-Empty and Compact

Proposition

If u is continuous, then $x^*(\mathbf{p}, w)$ is nonempty and compact.

- We already proved this as well.

Proof.

Define A by
$$A = B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$$

- This is a closed and bounded (i.e. compact, set) and
$$x^*(\mathbf{p}, w) = C_{\succsim}(A) = C_{\succsim}(B(\mathbf{p}, w))$$
where $\succsim$ are the preferences represented by u.
- Then $x^*(\mathbf{p}, w)$ is the set of maximizers of a continuous function over a compact set.

□

Walrasian Demand: Examples

How do we find Walrasian Demand?

- Need to solve a constrained maximization problem, usually using calculus.

Question 4, Problem Set 2; due next Wednesday.

For each of the following utility functions, find the Walrasian demand correspondence. (Hint: pictures may help)

1. $u(\mathbf{x}) = \prod_{i=1}^{n} x_i^{\alpha_i}$ with $\alpha_i > 0$ (generalized Cobb-Douglas).
2. $u(\mathbf{x}) = \min\{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n\}$ with $\alpha_i > 0$ (generalized Leontief).
3. $u(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i x_i$ for $\alpha_i > 0$ (generalized linear).
4. $u(\mathbf{x}) = \left[\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right]^{\frac{1}{\rho}}$ (generalized CES).

- Can we do the second one using calculus?
- How about the third? Do we need calculus?
- Constant elasticity of substitution (CES) preferences are the most commonly used homothetic preferences. Many preferences are a special case of CES.

An Optimization Recipe

How to solve $\max f(x)$ **subject to** $g_i(x) \leq 0$ **with** $i = 1, .., m$

1. Write the Langrange function $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ as

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

2. Write the First Order Conditions:

$$\underbrace{\overbrace{\nabla L(\mathbf{x}, \boldsymbol{\lambda})}^{n \times 1} = \nabla f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) = \mathbf{0}}_{\frac{\partial f(\mathbf{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\mathbf{x})}{\partial x_j} = 0 \text{ for all } j = 1,..,n}$$

3. Write constraints, inequalities for $\boldsymbol{\lambda}$, and complementary slackness conditions:

$$\begin{aligned} g_i(\mathbf{x}) &\leq 0 \quad \text{with} \quad i = 1, .., m \\ \lambda_i &\geq 0 \quad \text{with} \quad i = 1, .., m \\ \lambda_i g_i(\mathbf{x}) &= 0 \quad \text{with} \quad i = 1, .., m \end{aligned}$$

4. Find the $\mathbf{x}$ and $\boldsymbol{\lambda}$ that satisfy all these and you are done...hopefully.

The Recipe In Action: Cobb-Dougals Utility

Compute Walrasian demand when the utility function is $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

Here $x^*(\mathbf{p}, w)$ is the solution to

$$\max_{x_1, x_2 \in \{p_1 x_1 + p_2 x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0\}} x_1^{\alpha} x_2^{1-\alpha}$$

1. The Langrangian is

$$L(\mathbf{x}, \boldsymbol{\lambda}) = x_1^{\alpha} x_2^{1-\alpha} - \lambda_w (p_1 x_1 + p_2 x_2 - w) - (-\lambda_1 x_1) - (-\lambda_2 x_2)$$

2. The First Order Condition (w.r.t. $\mathbf{x}$) is:

$$\underbrace{\nabla L(\mathbf{x}, \boldsymbol{\lambda})}_{2\times 1} = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda_w p_1 + \lambda_1 \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda_w p_2 + \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha \frac{u(x_1, x_2)}{x_1} - \lambda_w p_1 + \lambda_1 \\ (1-\alpha) \frac{u(x_1, x_2)}{x_2} - \lambda_w p_2 + \lambda_2 \end{pmatrix} = \mathbf{0}$$

3. The constraints, inequalities for $\boldsymbol{\lambda}$, and complementary slackness are:

$$\begin{array}{cccc} p_1 x_1 + p_2 x_2 - w \leq 0 & -x_1 \leq 0, & \text{and} & -x_2 \leq 0 \\ \lambda_w \geq 0, & \lambda_1 \geq 0, & \text{and} & \lambda_2 \geq 0 \\ \lambda_w (p_1 x_1 + p_2 x_2 - w) = 0, & \lambda_1 x_1 = 0, & \text{and} & \lambda_2 x_2 = 0 \end{array}$$

4. Find a solution to the above (easy for me to say).

The Recipe In Action: Cobb-Dougals Utility

Compute Walrasian demand when the utility function is $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

We must solve:

$$\alpha \frac{u(x_1,x_2)}{x_1} - \lambda_w p_1 + \lambda_1 = 0 \quad \text{and} \quad (1-\alpha)\frac{u(x_1,x_2)}{x_2} - \lambda_w p_2 + \lambda_2 = 0$$

$$\begin{matrix} p_1 x_1 + p_2 x_2 - w \leq 0 \\ -x_1 \leq 0, \quad -x_2 \leq 0 \end{matrix} \quad \text{and} \quad \lambda_w \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0$$

$$\lambda_w (p_1 x_1 + p_2 x_2 - w) = 0 \quad \text{and} \quad \lambda_1 x_1 = 0, \ \lambda_2 x_2 = 0$$

- $\mathbf{x}^*(p, w)$ must be strictly positive (why?), hence $\lambda_1 = \lambda_2 = 0$.
- The budget constraint must bind (why?), hence $\lambda_w \geq 0$.
- Therefore the top two equalities become

$$\alpha u(x_1, x_2) = \lambda_w p_1 x_1 \qquad \text{and} \qquad (1-\alpha) u(x_1, x_2) = \lambda_w p_2 x_2$$

- Summing both sides and using Full Expenditure we get

$$u(x_1, x_2) = \lambda_w (p_1 x_1 + p_2 x_2) = \lambda_w w$$

- Substituting back then yields

$$x_1^*(p, w) = \frac{\alpha w}{p_1}, \ x_2^*(p, w) = \frac{(1-\alpha) w}{p_2}, \text{ and } \lambda_w = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$$

Next Class

- More Properties of Walrasian Demand.
- Indirect Utility.
- Comparative Statics.
- Expenditure Minimization.